

# Remarks on the neutrino oscillation formula

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## Abstract

We show that the neutrino oscillation formula recently derived in the quantum field theory framework holds true despite the arbitrariness in the mass parameter for the flavor fields. This formula is exact and exhibits new features with respect to the usual Pontecorvo formula, which is however valid in the relativistic limit.

P.A.C.S.: 14.60.Pq, 11.15.Tk, 12.15.Ff

## I. INTRODUCTION

In view of the great theoretical and experimental interest [1] several papers have been recently devoted to the quantum field theoretical approach to the neutrino mixing and oscillations [2–5]. In particular, the study of the generator of the Pontecorvo mixing transformations [6] has shown [3] that the Hilbert space where the mixed (flavor) field operators are defined is unitarily inequivalent, in the infinite volume limit, to the Hilbert space for the original (unmixed) field operators. Such a finding leads to a novel understanding of the field mixing and to a new, exact oscillation formula [4], which reduces to the Pontecorvo one in the relativistic limit.

As a matter of fact, the problem of the definition and of the physical interpretation of the state space for the flavor fields is a controversial one and it is the object of still open discussions [2–5]. On the other hand, the discovery [3] that the Pontecorvo field mixing transformation is a non-unitarily implementable transformation rests on firm mathematical grounds, so that it cannot be ignored in any discussion on the field mixing problems. In a recent paper [5] it has been thus considered the degree of arbitrariness involved in the construction of the flavor states starting from the results of refs. [3,4] (denoted by the authors of ref. [5] as the BV formalism).

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By using a more general mass parameterization than the one adopted in the BV formalism, the conclusion of the authors of ref. [5] has been that, since the mass arbitrariness shows up in the oscillation probability, the conclusions drawn in refs. [3] are unphysical. In particular, they conclude that the procedure adopted in ref. [3] by choosing a specific mass parameterization has no physical basis. They then examine the structure of the neutrino propagator. However, although the physical poles of the propagator are shown to coincide with the eigenvalues of the mass matrix in the Lagrangian and appear not to be affected by the mass parameterization arbitrariness, they are not able to escape the arbitrariness difficulty in the oscillation formula. Thus they are finally forced to admit that it is still necessary to investigate in detail how to derive oscillations formulas reflecting real experimental situations on the basis of the field theory.

Motivated by such a necessity, in the present paper we consider the arguments of ref. [5] and we show that, even by applying the more general mass parameterization there adopted, the mass arbitrariness disappears from the exact oscillation formula as derived in ref. [4]. Actually, the oscillation formula considered in ref. [5] is the approximate one which was derived in ref. [3], and not its exact form obtained in the Green's function formalism of ref. [4].

The conclusions of ref. [5] about the unphysical basis of the analysis or refs. [3,4] are thus ruled out. On the contrary, even though the problem of the interpretation of the flavor space may still be object of discussion, the field theoretical formalism derived in refs. [3,4] appears to be sound and, what is most interesting, it leads to the oscillation formula which is experimentally testable.

In the present paper we also shortly comment on the physical meaning of the mass arbitrariness which has been introduced by the authors of ref. [5] without explicit justification. These comments are also useful in order to clarify the physical meaning of the more general transformations used in ref. [5] and of the particular choice adopted in the BV formalism.

The paper is organized as follows. In Section II generalities on the formalism are reviewed, also considering the generalization of ref. [5]. In Section III the exact oscillation formula is rederived in the formalism of ref. [5] and it is shown to be independent of the mass arbitrariness. Section IV is devoted to further comments and conclusions.

## II. MIXED FERMIONS

To be definite, let us consider an oversimplified model with two (Dirac) fermion fields with a mixed mass term:

$$\mathcal{L} = \bar{\nu}_e (i \not{\partial} - m_e) \nu_e + \bar{\nu}_\mu (i \not{\partial} - m_\mu) \nu_\mu - m_{e\mu} (\bar{\nu}_e \nu_\mu + \bar{\nu}_\mu \nu_e) . \quad (1)$$

The above Lagrangian is sufficient in order to describe the single-particle evolution of a mixed state, i.e. neutrino oscillations [4], and can be fully diagonalized by the transformation

$$\begin{aligned} \nu_e(x) &= \nu_1(x) \cos \theta + \nu_2(x) \sin \theta \\ \nu_\mu(x) &= -\nu_1(x) \sin \theta + \nu_2(x) \cos \theta , \end{aligned} \quad (2)$$

where  $\theta$  is the mixing angle and  $m_e = m_1 \cos^2 \theta + m_2 \sin^2 \theta$ ,  $m_\mu = m_1 \sin^2 \theta + m_2 \cos^2 \theta$ ,  $m_{e\mu} = (m_2 - m_1) \sin \theta \cos \theta$ .  $\nu_1$  and  $\nu_2$  therefore are non-interacting, free fields, anticommuting with each other at any space-time point. They are explicitly given by

$$\nu_i(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}, r} \left[ u_{\mathbf{k}, i}^r \alpha_{\mathbf{k}, i}^r(t) + v_{-\mathbf{k}, i}^r \beta_{-\mathbf{k}, i}^{r\dagger}(t) \right] e^{i\mathbf{k} \cdot \mathbf{x}}, \quad i = 1, 2. \quad (3)$$

with  $\alpha_{\mathbf{k}, i}^r(t) = e^{-i\omega_i t} \alpha_{\mathbf{k}, i}^r(0)$ ,  $\beta_{\mathbf{k}, i}^r(t) = e^{-i\omega_i t} \beta_{\mathbf{k}, i}^r(0)$  and  $\omega_i = \sqrt{\mathbf{k}^2 + m_i^2}$ . Here and in the following we use  $t \equiv x_0$ , when no misunderstanding arises. The vacuum for the  $\alpha_i$  and  $\beta_i$  operators is denoted by  $|0\rangle_{1,2}$ :  $\alpha_{\mathbf{k}, i}^r |0\rangle_{1,2} = \beta_{\mathbf{k}, i}^r |0\rangle_{1,2} = 0$ . The anticommutation relations are the usual ones (see ref. [3]). The orthonormality and completeness relations are:

$$u_{\mathbf{k}, i}^{r\dagger} u_{\mathbf{k}, i}^s = v_{\mathbf{k}, i}^{r\dagger} v_{\mathbf{k}, i}^s = \delta_{rs}, \quad u_{\mathbf{k}, i}^{r\dagger} v_{-\mathbf{k}, i}^s = v_{-\mathbf{k}, i}^{r\dagger} u_{\mathbf{k}, i}^s = 0, \quad \sum_r (u_{\mathbf{k}, i}^r u_{\mathbf{k}, i}^{r\dagger} + v_{-\mathbf{k}, i}^r v_{-\mathbf{k}, i}^{r\dagger}) = I. \quad (4)$$

The fields  $\nu_e$  and  $\nu_\mu$  are thus completely determined through eq.(2), which can be rewritten in the following form (we use  $(\sigma, j) = (e, 1), (\mu, 2)$ ):

$$\nu_\sigma(x) = G_\theta^{-1}(t) \nu_j(x) G_\theta(t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}, r} \left[ u_{\mathbf{k}, j}^r \alpha_{\mathbf{k}, \sigma}^r(t) + v_{-\mathbf{k}, j}^r \beta_{-\mathbf{k}, \sigma}^{r\dagger}(t) \right] e^{i\mathbf{k} \cdot \mathbf{x}}, \quad (5)$$

$$G_\theta(t) = \exp \left[ \theta \int d^3 \mathbf{x} \left( \nu_1^\dagger(x) \nu_2(x) - \nu_2^\dagger(x) \nu_1(x) \right) \right] \quad (6)$$

where  $G_\theta(t)$  is the generator of the mixing transformations (2) (see ref. [3] for a discussion of its properties).

Eq.(5) gives an expansion of the flavor fields  $\nu_e$  and  $\nu_\mu$  in the same basis of  $\nu_1$  and  $\nu_2$ . In the BV formalism the flavor annihilation operators are then identified with

$$\begin{pmatrix} \alpha_{\mathbf{k}, \sigma}^r(t) \\ \beta_{-\mathbf{k}, \sigma}^{r\dagger}(t) \end{pmatrix} = G_\theta^{-1}(t) \begin{pmatrix} \alpha_{\mathbf{k}, j}^r(t) \\ \beta_{-\mathbf{k}, j}^{r\dagger}(t) \end{pmatrix} G_\theta(t) \quad (7)$$

The BV flavor vacuum is defined as  $|0(t)\rangle_{e,\mu} \equiv G_\theta^{-1}(t) |0\rangle_{1,2}$ .

The explicit expression of the flavor annihilation operators is (we choose  $\mathbf{k} = (0, 0, |\mathbf{k}|)$ ):

$$\begin{pmatrix} \alpha_{\mathbf{k}, e}^r(t) \\ \alpha_{\mathbf{k}, \mu}^r(t) \\ \beta_{-\mathbf{k}, e}^{r\dagger}(t) \\ \beta_{-\mathbf{k}, \mu}^{r\dagger}(t) \end{pmatrix} = \begin{pmatrix} c_\theta & s_\theta |U_{\mathbf{k}}| & 0 & s_\theta \epsilon^r |V_{\mathbf{k}}| \\ -s_\theta |U_{\mathbf{k}}| & c_\theta & s_\theta \epsilon^r |V_{\mathbf{k}}| & 0 \\ 0 & -s_\theta \epsilon^r |V_{\mathbf{k}}| & c_\theta & s_\theta |U_{\mathbf{k}}| \\ -s_\theta \epsilon^r |V_{\mathbf{k}}| & 0 & -s_\theta |U_{\mathbf{k}}| & c_\theta \end{pmatrix} \begin{pmatrix} \alpha_{\mathbf{k}, 1}^r(t) \\ \alpha_{\mathbf{k}, 2}^r(t) \\ \beta_{-\mathbf{k}, 1}^{r\dagger}(t) \\ \beta_{-\mathbf{k}, 2}^{r\dagger}(t) \end{pmatrix} \quad (8)$$

where  $c_\theta \equiv \cos \theta$ ,  $s_\theta \equiv \sin \theta$ ,  $\epsilon^r \equiv (-1)^r$ , and

$$|U_{\mathbf{k}}| \equiv u_{\mathbf{k}, 2}^{r\dagger} u_{\mathbf{k}, 1}^r = v_{-\mathbf{k}, 1}^{r\dagger} v_{-\mathbf{k}, 2}^r \quad (9)$$

$$|V_{\mathbf{k}}| \equiv \epsilon^r u_{\mathbf{k}, 1}^{r\dagger} v_{-\mathbf{k}, 2}^r = -\epsilon^r u_{\mathbf{k}, 2}^{r\dagger} v_{-\mathbf{k}, 1}^r. \quad (10)$$

We have:

$$|U_{\mathbf{k}}| = \left( \frac{\omega_{k,1} + m_1}{2\omega_{k,1}} \right)^{\frac{1}{2}} \left( \frac{\omega_{k,2} + m_2}{2\omega_{k,2}} \right)^{\frac{1}{2}} \left( 1 + \frac{|\mathbf{k}|^2}{(\omega_{k,1} + m_1)(\omega_{k,2} + m_2)} \right) \quad (11)$$

$$|V_{\mathbf{k}}| = \left( \frac{\omega_{k,1} + m_1}{2\omega_{k,1}} \right)^{\frac{1}{2}} \left( \frac{\omega_{k,2} + m_2}{2\omega_{k,2}} \right)^{\frac{1}{2}} \left( \frac{\mathbf{k}}{(\omega_{k,2} + m_2)} - \frac{\mathbf{k}}{(\omega_{k,1} + m_1)} \right) \quad (12)$$

$$|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1 \quad (13)$$

It has been recently noticed [5], however, that expanding the flavor fields in the same basis as the (free) fields with definite masses is actually a special choice, and that a more general possibility exists.

In other words, in the expansion eq.(5) one could use eigenfunctions with arbitrary masses  $\mu_\sigma$ , and therefore not necessarily the same as the masses which appear in the Lagrangian. On this basis, the authors of ref. [5] have generalized the BV transformation (7) by writing the flavor fields as

$$\nu_\sigma(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}, r} \left[ u_{\mathbf{k}, \sigma}^r \tilde{\alpha}_{\mathbf{k}, \sigma}^r(t) + v_{-\mathbf{k}, \sigma}^r \tilde{\beta}_{-\mathbf{k}, \sigma}^{r\dagger}(t) \right] e^{i\mathbf{k} \cdot \mathbf{x}}, \quad (14)$$

where  $u_\sigma$  and  $v_\sigma$  are the helicity eigenfunctions with mass  $\mu_\sigma$ <sup>1</sup>. We denote by a tilde the generalized flavor operators introduced in ref. [5] in order to distinguish them from the ones defined in eq.(7). The expansion eq.(14) is more general than the one in eq.(5) since the latter corresponds to the particular choice  $\mu_e \equiv m_1$ ,  $\mu_\mu \equiv m_2$ .

The relation between the flavor and the mass operators is now:

$$\begin{pmatrix} \tilde{\alpha}_{\mathbf{k}, \sigma}^r(t) \\ \tilde{\beta}_{-\mathbf{k}, \sigma}^{r\dagger}(t) \end{pmatrix} = K_{\theta, \mu}^{-1}(t) \begin{pmatrix} \alpha_{\mathbf{k}, j}^r(t) \\ \beta_{-\mathbf{k}, j}^{r\dagger}(t) \end{pmatrix} K_{\theta, \mu}(t) \quad , \quad (15)$$

with  $(\sigma, j) = (e, 1), (\mu, 2)$  and where  $K_{\theta, \mu}(t)$  is the generator of the transformation (2) and can be written as

$$K_{\theta, \mu}(t) = I_\mu(t) G_\theta(t) \quad (16)$$

$$I_\mu(t) = \prod_{\mathbf{k}, r} \exp \left\{ i \sum_{(\sigma, j)} \xi_{\sigma, j}^{\mathbf{k}} \left[ \alpha_{\mathbf{k}, j}^{r\dagger}(t) \beta_{-\mathbf{k}, j}^{r\dagger}(t) + \beta_{-\mathbf{k}, j}^r(t) \alpha_{\mathbf{k}, j}^r(t) \right] \right\} \quad (17)$$

with  $\xi_{\sigma, j}^{\mathbf{k}} \equiv (\chi_\sigma - \chi_j)/2$  and  $\cot \chi_\sigma = |\mathbf{k}|/\mu_\sigma$ ,  $\cot \chi_j = |\mathbf{k}|/m_j$ . For  $\mu_e \equiv m_1$ ,  $\mu_\mu \equiv m_2$  one has  $I_\mu(t) = 1$ .

The explicit matrix form of the flavor operators is [5]:

$$\begin{pmatrix} \tilde{\alpha}_{\mathbf{k}, e}^r(t) \\ \tilde{\alpha}_{\mathbf{k}, \mu}^r(t) \\ \tilde{\beta}_{-\mathbf{k}, e}^{r\dagger}(t) \\ \tilde{\beta}_{-\mathbf{k}, \mu}^{r\dagger}(t) \end{pmatrix} = \begin{pmatrix} c_\theta \rho_{e1}^{\mathbf{k}} & s_\theta \rho_{e2}^{\mathbf{k}} & i c_\theta \lambda_{e1}^{\mathbf{k}} & i s_\theta \lambda_{e2}^{\mathbf{k}} \\ -s_\theta \rho_{\mu1}^{\mathbf{k}} & c_\theta \rho_{\mu2}^{\mathbf{k}} & -i s_\theta \lambda_{\mu1}^{\mathbf{k}} & i c_\theta \lambda_{\mu2}^{\mathbf{k}} \\ i c_\theta \lambda_{e1}^{\mathbf{k}} & i s_\theta \lambda_{e2}^{\mathbf{k}} & c_\theta \rho_{e1}^{\mathbf{k}} & s_\theta \rho_{e2}^{\mathbf{k}} \\ -i s_\theta \lambda_{\mu1}^{\mathbf{k}} & i c_\theta \lambda_{\mu2}^{\mathbf{k}} & -s_\theta \rho_{\mu1}^{\mathbf{k}} & c_\theta \rho_{\mu2}^{\mathbf{k}} \end{pmatrix} \begin{pmatrix} \alpha_{\mathbf{k}, 1}^r(t) \\ \alpha_{\mathbf{k}, 2}^r(t) \\ \beta_{-\mathbf{k}, 1}^{r\dagger}(t) \\ \beta_{-\mathbf{k}, 2}^{r\dagger}(t) \end{pmatrix} \quad (18)$$

where  $c_\theta \equiv \cos \theta$ ,  $s_\theta \equiv \sin \theta$  and

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<sup>1</sup>The use of such a basis simplifies considerably calculations with respect to the original choice of ref. [3].

$$\rho_{ab}^{\mathbf{k}} \delta_{rs} \equiv \cos \frac{\chi_a - \chi_b}{2} \delta_{rs} = u_{\mathbf{k},a}^{r\dagger} u_{\mathbf{k},b}^s = v_{-\mathbf{k},a}^{r\dagger} v_{-\mathbf{k},b}^s \quad (19)$$

$$i\lambda_{ab}^{\mathbf{k}} \delta_{rs} \equiv i \sin \frac{\chi_a - \chi_b}{2} \delta_{rs} = u_{\mathbf{k},a}^{r\dagger} v_{-\mathbf{k},b}^s = v_{-\mathbf{k},a}^{r\dagger} u_{\mathbf{k},b}^s \quad (20)$$

with  $a, b = 1, 2, e, \mu$ . Since  $\rho_{12}^{\mathbf{k}} = |U_{\mathbf{k}}|$  and  $i\lambda_{12}^{\mathbf{k}} = \epsilon^r |V_{\mathbf{k}}|$ , etc., the operators (18) reduce to the ones in eqs.(8) when  $\mu_e \equiv m_1$  and  $\mu_\mu \equiv m_2$ <sup>2</sup>.

Finally, the generalization of the BV flavor vacuum, which is annihilated by the flavor operators given by eq.(15), is now written as [5]:

$$|\tilde{0}(t)\rangle_{e,\mu} \equiv K_{\theta,\mu}^{-1}(t)|0\rangle_{1,2} \quad (21)$$

For  $\mu_e \equiv m_1$  and  $\mu_\mu \equiv m_2$ , this state reduces to the BV flavor vacuum  $|0(t)\rangle_{e,\mu}$  above defined.

### III. THE OSCILLATION FORMULA

In the formal framework of the previous Section the annihilation operators and the vacuum for mixed fermions are defined self-consistently, and the Hilbert space for mixed neutrinos can thus be constructed. Such an Hilbert space, however, has built in the arbitrariness related with the mass parameters  $\mu_\sigma$ ,  $\sigma = e, \mu$ . According to the authors of ref. [5], such an arbitrariness also shows up in the final expression of the oscillation probability, which is a non-acceptable result since the theory arbitrary parameters should have no effects on observable quantities. The full construction, although mathematically consistent, would then be questionable from a physical point of view. Now we show that the analysis of ref. [5] is not complete, that the exact oscillation probabilities are independent of the arbitrary mass parameters and therefore the conclusion of ref. [5] is ruled out.

The main point is that the authors of ref. [5] miss to compute the full oscillation probability whose exact form is presented in ref. [4] and it is there obtained in the Green's function formalism. In fact, the statement of ref. [5] that the oscillation formula “seems not to be correct” since it is based on the one neutrino state, which does depend on  $\mu_e$  and  $\mu_\mu$ , is not correct: as we show below, it is possible to calculate the oscillation probabilities by using the arbitrary mass formalism of Section II, getting a result which is independent of the arbitrary masses  $\mu_e$  and  $\mu_\mu$  and coincides with the one of ref. [4].

In the line of ref. [4], let us consider the propagator for the flavor fields, which has to be defined on the proper (flavor) vacuum. Notice that here we perform the computations in the generalized BV formalism of ref. [5]. The propagators are then given by:

$$\begin{pmatrix} \tilde{G}_{ee}(x, y) & \tilde{G}_{\mu e}(x, y) \\ \tilde{G}_{e\mu}(x, y) & \tilde{G}_{\mu\mu}(x, y) \end{pmatrix} \equiv {}_{e,\mu}\langle \tilde{0}(y_0) | \begin{pmatrix} T[\nu_e(x)\bar{\nu}_e(y)] & T[\nu_\mu(x)\bar{\nu}_e(y)] \\ T[\nu_e(x)\bar{\nu}_\mu(y)] & T[\nu_\mu(x)\bar{\nu}_\mu(y)] \end{pmatrix} | \tilde{0}(y_0) \rangle_{e,\mu}, \quad (22)$$

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<sup>2</sup>In performing such an identification, one should take into account that the operators for antiparticles differ for a minus sign, related to the different spinor bases used in the expansions (5) and (14). Such a sign difference is however irrelevant in what follows.

where the state used is the one defined in eq.(21). These propagators clearly *do depend* on the arbitrary parameters  $\mu_e$  and  $\mu_\mu$ , which are present in  $|\tilde{0}(y_0)\rangle_{e,\mu}$ . However, the propagator is not a measurable quantity: on the contrary, the oscillation probability, which can be defined in terms of it, is measurable and should not be affected by any arbitrary parameters.

Let us then consider the case of an initial electron neutrino which evolves (oscillates) in time. The two relevant propagators are [4]:

$$i\tilde{G}_{ee}^>(t, \mathbf{x}; 0, \mathbf{y}) = {}_{e,\mu}\langle \tilde{0} | \nu_e(t, \mathbf{x}) \bar{\nu}_e(0, \mathbf{y}) | \tilde{0} \rangle_{e,\mu} \quad (23)$$

$$i\tilde{G}_{\mu e}^>(t, \mathbf{x}; 0, \mathbf{y}) = {}_{e,\mu}\langle \tilde{0} | \nu_\mu(t, \mathbf{x}) \bar{\nu}_e(0, \mathbf{y}) | \tilde{0} \rangle_{e,\mu} \quad (24)$$

where  $|\tilde{0}\rangle_{e,\mu} \equiv |\tilde{0}(t=0)\rangle_{e,\mu}$ . As discussed in [4], there are four distinct transition amplitudes which can be defined from the above propagators. In the present generalized case, we have to use the wave functions  $u_\sigma$  and  $v_\sigma$ , instead of  $u_j$  and  $v_j$  which were used in [4]. This is in line with the discussion of the previous Section: the choice of the basis in which we expand the flavor fields determines the relevant annihilators and then the vacuum.

The amplitudes are then obtained as<sup>3</sup>

$$\tilde{\mathcal{P}}_{ee}^r(\mathbf{k}, t) \equiv i u_{\mathbf{k},e}^{r\dagger} \tilde{G}_{ee}^>(\mathbf{k}, t) \gamma^0 u_{\mathbf{k},e}^r = \left\{ \tilde{\alpha}_{\mathbf{k},e}^r(t), \tilde{\alpha}_{\mathbf{k},e}^{r\dagger}(0) \right\} \quad (25)$$

$$\tilde{\mathcal{P}}_{\bar{e}e}^r(\mathbf{k}, t) \equiv i v_{-\mathbf{k},e}^{r\dagger} \tilde{G}_{ee}^>(\mathbf{k}, t) \gamma^0 u_{\mathbf{k},e}^r = \left\{ \tilde{\beta}_{-\mathbf{k},e}^{r\dagger}(t), \tilde{\alpha}_{\mathbf{k},e}^{r\dagger}(0) \right\} \quad (26)$$

$$\tilde{\mathcal{P}}_{\mu e}^r(\mathbf{k}, t) \equiv i u_{\mathbf{k},\mu}^{r\dagger} \tilde{G}_{\mu e}^>(\mathbf{k}, t) \gamma^0 u_{\mathbf{k},e}^r = \left\{ \tilde{\alpha}_{\mathbf{k},\mu}^r(t), \tilde{\alpha}_{\mathbf{k},e}^{r\dagger}(0) \right\} \quad (27)$$

$$\tilde{\mathcal{P}}_{\bar{\mu}e}^r(\mathbf{k}, t) \equiv i v_{-\mathbf{k},\mu}^{r\dagger} \tilde{G}_{\mu e}^>(\mathbf{k}, t) \gamma^0 u_{\mathbf{k},e}^r = \left\{ \tilde{\beta}_{-\mathbf{k},\mu}^{r\dagger}(t), \tilde{\alpha}_{\mathbf{k},e}^{r\dagger}(0) \right\} \quad (28)$$

The explicit form of these amplitudes is rather complicated. Notice that all of them involve the arbitrary parameters  $\mu_\sigma$ . However, it can be verified that the following sum rule for the squared moduli is still valid:

$$\left| \tilde{\mathcal{P}}_{ee}^r(\mathbf{k}, t) \right|^2 + \left| \tilde{\mathcal{P}}_{\bar{e}e}^r(\mathbf{k}, t) \right|^2 + \left| \tilde{\mathcal{P}}_{\mu e}^r(\mathbf{k}, t) \right|^2 + \left| \tilde{\mathcal{P}}_{\bar{\mu}e}^r(\mathbf{k}, t) \right|^2 = 1, \quad (29)$$

Moreover, tediously long but substantially simple algebra shows that

$$\left| \left\{ \tilde{\alpha}_{\mathbf{k},e}^r(t), \tilde{\alpha}_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2 + \left| \left\{ \tilde{\beta}_{-\mathbf{k},e}^{r\dagger}(t), \tilde{\alpha}_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2 = \left| \left\{ \alpha_{\mathbf{k},e}^r(t), \alpha_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2 + \left| \left\{ \beta_{-\mathbf{k},e}^{r\dagger}(t), \alpha_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2 \quad (30)$$

$$\left| \left\{ \tilde{\alpha}_{\mathbf{k},\mu}^r(t), \tilde{\alpha}_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2 + \left| \left\{ \tilde{\beta}_{-\mathbf{k},\mu}^{r\dagger}(t), \tilde{\alpha}_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2 = \left| \left\{ \alpha_{\mathbf{k},\mu}^r(t), \alpha_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2 + \left| \left\{ \beta_{-\mathbf{k},\mu}^{r\dagger}(t), \alpha_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2 \quad (31)$$

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<sup>3</sup>With respect to the ones defined in ref. [4], we omit here an (irrelevant) phase factor. This is due to the different definition of the flavor operators – see eq.(8).

which is the announced result. In fact in ref. [4] the probabilities for oscillating neutrinos were found to be

$$P_{\nu_e \rightarrow \nu_e}(\mathbf{k}, t) = \left| \left\{ \alpha_{\mathbf{k},e}^r(t), \alpha_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2 + \left| \left\{ \beta_{-\mathbf{k},e}^{r\dagger}(t), \alpha_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2 \quad (32)$$

$$= 1 - \sin^2(2\theta) \left[ |U_{\mathbf{k}}|^2 \sin^2 \left( \frac{\omega_{k,2} - \omega_{k,1}}{2} t \right) + |V_{\mathbf{k}}|^2 \sin^2 \left( \frac{\omega_{k,2} + \omega_{k,1}}{2} t \right) \right],$$

$$P_{\nu_e \rightarrow \nu_\mu}(\mathbf{k}, t) = \left| \left\{ \alpha_{\mathbf{k},\mu}^r(t), \alpha_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2 + \left| \left\{ \beta_{-\mathbf{k},\mu}^{r\dagger}(t), \alpha_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2 \quad (33)$$

$$= \sin^2(2\theta) \left[ |U_{\mathbf{k}}|^2 \sin^2 \left( \frac{\omega_{k,2} - \omega_{k,1}}{2} t \right) + |V_{\mathbf{k}}|^2 \sin^2 \left( \frac{\omega_{k,2} + \omega_{k,1}}{2} t \right) \right].$$

Thus, we have proven that the generalized formalism of ref. [5] leads to the same exact result of ref. [4]. The above formula is still valid in the more general case: the probabilities for oscillating neutrinos do not depend on any arbitrary mass parameters.

#### IV. COMMENTS AND CONCLUSION

In ref. [5] the flavor field wave functions  $u_\sigma$  and  $v_\sigma$  have been introduced which satisfy the free Dirac equations with arbitrary mass  $\mu_\sigma$  ( $\sigma = e, \mu$ ). Its introduction has not been justified in ref. [5]. Therefore a short comment about the physical meaning of such a procedure may be in order and it can be also useful for a better understanding of the formalism. Use of the wave functions  $u_\sigma$  and  $v_\sigma$  clearly represents a more general choice than the one made in the BV formalism [3], where  $\mu_e \equiv m_1$ ,  $\mu_\mu \equiv m_2$  has been used.

We observe that the mass parameter  $\mu_\sigma$  represents the “bare” mass of the corresponding field and therefore it can be given any arbitrary value. Moreover, for  $\theta = 0$  the transformation (15) reduces to the transformation generated by  $I_\mu(t)$  given by eq.(17): now note that this is nothing but a Bogoliubov transformation which, at  $\theta = 0$ , relates unmixed field operators,  $\alpha_j$  and, say,  $a_j(\xi_{\sigma,j})$ , of masses  $m_j$  and  $\mu_\sigma$ , respectively. In the language of LSZ formalism of QFT [7,8], the  $\alpha_j$  refer to physical (free) fields; the  $a_j(\xi_{\sigma,j})$  to Heisenberg (interacting) fields. In the infinite volume limit, the Hilbert spaces where the operators  $\alpha_j$  and  $a_j$  are respectively defined, turn out to be unitarily inequivalent spaces. Moreover, the transformation parameter  $\xi_{\sigma,j}$  acts as a label specifying Hilbert spaces unitarily inequivalent among themselves (for each (different) value of the  $\mu_\sigma$  mass parameter). The crucial point is that the physically relevant space is the one associated with the observable physical mass  $m_j$ , the other ones being associated with the bare masses  $\mu_\sigma$ ’s. It can be shown [7] that the masses  $\mu_\sigma$  dynamically acquire a convenient mass shift term such that the asymptotic physical  $\alpha_j$ -fields are associated with the physical mass  $m_j$  and the arbitrariness intrinsic to the bare mass  $\mu_\sigma$  does not affect the observables.

Therefore, in principle any one of the  $\xi$ -parameterized Hilbert spaces can be chosen to work with (in other words, the bare masses can be given any arbitrary value). Since, however, one is interested in observable quantities, in the LSZ formalism [7,8] the space one chooses to work with is the free physical field space (associated to the  $\alpha_j$  operator fields, in our case). This is the “particular” choice made in the BV formalism. In the generalized BV formalism instead, by means of the Bogoliubov transformation explicitly given by eq.(18)

written for  $\theta = 0$ , one first moves to the operators  $a_j(\xi_{\sigma,j})$ , leaving the  $\xi$  value unspecified (i.e. for arbitrary mass parameter  $\mu_\sigma$ ) and then one considers the mixing problem. Of course, at the end of the computations observable quantities should not depend on the arbitrary parameters, as indeed in this paper we have proven it happens to be.

Here we are not going to give more details on the multiplicity of Hilbert spaces associated with arbitrary bare mass parameters. However, the above comment sheds some light on the physical meaning of the particular choice made in refs. [3,4], and it also suggests to us why the result of the computations presented in the present paper actually was to be expected on a physical ground, besides being supported by straightforward mathematics.

## ACKNOWLEDGEMENTS

This work was partially supported by MURST, INFN, and ESF.



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